

TRANSCENDENTAL ENDING LAMINATIONS

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ABSTRACT. Yair Minsky showed that punctured torus groups are classified by a pair of ending laminations (ν_-, ν_+) . In this note, we show that there are ending laminations ν_+ such that for any choice of ν_- , the punctured torus group is transcendental as a subgroup of $\mathrm{PSL}_2\mathbb{C}$.

1. INTRODUCTION

We investigate in this note the algebraic properties of Kleinian groups, that is finitely generated, discrete subgroups of $\mathrm{PSL}_2\mathbb{C}$. Let $\mathbb{A} \subset \mathbb{C}$ be the algebraic numbers. A subgroup $G < \mathrm{PSL}_2\mathbb{C}$ is *algebraic* if it can be conjugated into $\mathrm{PSL}_2\mathbb{A}$. Otherwise, it is *transcendental*. By Mostow rigidity, all finite covolume Kleinian groups are algebraic. In fact, it follows from the proof of Thurston's hyperbolization theorem for Haken manifolds that any Kleinian group has *some* faithful representation into $\mathrm{PSL}_2\mathbb{C}$ whose image is algebraic. This is proven by embedding the abstract Kleinian group into the fundamental group of a finite volume hyperbolic 3-orbifold. The induced hyperbolic structure on the subgroup from Thurston's construction is geometrically finite [Mo84]. It seems possible that there are algebraic Kleinian groups which are geometrically finite, but which are not the subgroup of any finite volume Kleinian group. However, it follows from the geometric tameness theorem [A04, CG04] and the covering theorem [C96] that if a non-geometrically finite Kleinian group Γ is a subgroup of a finite volume Kleinian group Γ' , then Γ is a surface group containing a fiber subgroup Γ_0 of a virtual fibration of \mathbb{H}^3/Γ' of index at most 2 in Γ . Each end of a Kleinian group is either geometrically finite, or simply degenerate [C93]. Thus, in this case Γ_0 has two simply degenerate ends, and there is an infinite order isometry $\psi \in \Gamma' - \Gamma_0$ which normalizes Γ_0 , and acts by a translation on \mathbb{H}^3/Γ_0 . It does not appear to be known whether there are degenerate algebraic Kleinian groups which are not of this type. A similar argument to the covering theorem shows that a singly degenerate hyperbolic 3-manifold cannot immerse totally geodesically in a higher dimensional geometrically finite hyperbolic manifold. Any algebraic Kleinian group Γ embeds in a natural way into a higher rank S-arithmetic lattice. But the embedding does not induce a quasi-isometric embedding on the convex core of \mathbb{H}^3/Γ into the corresponding cover of the symmetric space, otherwise the techniques of the covering theorem would carry over. One may construct simplicial ruled surfaces in higher rank symmetric spaces, but the bounded diameter lemma and local finiteness fail.

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It would be interesting to know if there can be singly degenerate algebraic surface Kleinian groups. We demonstrate in this note that there are families of degenerate punctured torus Kleinian groups satisfying certain geometric restrictions which must be transcendental. The families of groups we consider are uncountable, and therefore all but countably many groups in a family will be transcendental; the point of the result is that *all* of the groups in the family are transcendental.

One motivation for studying this question is to understand whether there is an algorithm to tell that a collection of matrices in $\mathrm{SL}_2\mathbb{A}$ generates a discrete subgroup of $\mathrm{SL}_2\mathbb{C}$. If every non-geometrically finite Kleinian group were the subgroup of a finite volume Kleinian group, then an algorithm would exist. So the discovery of a singly degenerate algebraic Kleinian group would demonstrate the difficulty of this question.

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2. ENDING LAMINATION THEOREM

In this section, we review the classification of punctured torus Kleinian groups from [M99]. Let Σ be a punctured torus. A punctured torus group G is a 2 generator free discrete subgroup of $\mathrm{PSL}_2\mathbb{C}$, such that there exist two generators of G whose commutator is parabolic. G is naturally identified as the image of a discrete faithful representation $\rho : \pi_1(\Sigma) \rightarrow \mathrm{PSL}_2\mathbb{C}$ such that the monodromy of the puncture is parabolic. Then $N = \mathbb{H}^3/G$ is a punctured torus manifold. Bonahon showed that $N \cong \Sigma \times \mathbb{R}$ [Bon86].

Let \mathbb{H}^2 denote the upper half plane model for hyperbolic space, with boundary \mathbb{R} . The Farey triangulation is a graph in $\overline{\mathbb{H}^2} = \mathbb{H}^2 \cup \mathbb{R} \cup \infty$ with vertices at $\hat{\mathbb{Q}} = \mathbb{Q} \cup \infty$, and two rational numbers $\frac{p}{q}, \frac{r}{s}$ are connected by a hyperbolic geodesic if $|ps - qr| = 1$ (where we let $\infty = \frac{1}{0}$). The irrational numbers in \mathbb{R} represent the ending laminations. See figure 1 for a partial picture of the Farey triangulation in the disk model of hyperbolic space. The curve complex $\mathcal{C}(\Sigma)$ of the punctured torus has vertices consisting of isotopy classes of simple closed curves. The edges of $\mathcal{C}(\Sigma)$ join pairs of curves which intersect exactly once. There is a natural bijection of $\mathcal{C}(\Sigma)^{(1)}$ with the Farey triangulation, for example by considering projective homology classes in $H_1(\Sigma) \cong \mathbb{Z}^2$. Given a basis $\{\alpha, \beta\}$ for $H_1(\Sigma)$, any oriented simple closed curve represents a homology class $p\alpha + q\beta$, with $\gcd(p, q) = 1$. Then we get an element $-\frac{p}{q} \in \hat{\mathbb{Q}}$, which gives a bijection with unoriented simple closed curves. The number $|pr - qs|$ represents the geometric intersection number between two simple closed curves $p/q, r/s$.

If we lift a punctured torus group representation $\rho : \pi_1\Sigma \rightarrow \mathrm{PSL}_2\mathbb{C}$ to $\tilde{\rho} : \pi_1\Sigma \rightarrow \mathrm{SL}_2\mathbb{C}$, then the function $\mathrm{tr}(\tilde{\rho}(\alpha))$ is well-defined on conjugacy classes of elements in $\pi_1\Sigma$. This function takes values in \mathbb{A} if ρ is algebraic. The lift $\tilde{\rho}$ is well-defined up to multiplying generators by $-I$. Given three adjacent curves $\alpha, \beta, \gamma \in \mathcal{C}(\Sigma)$, with

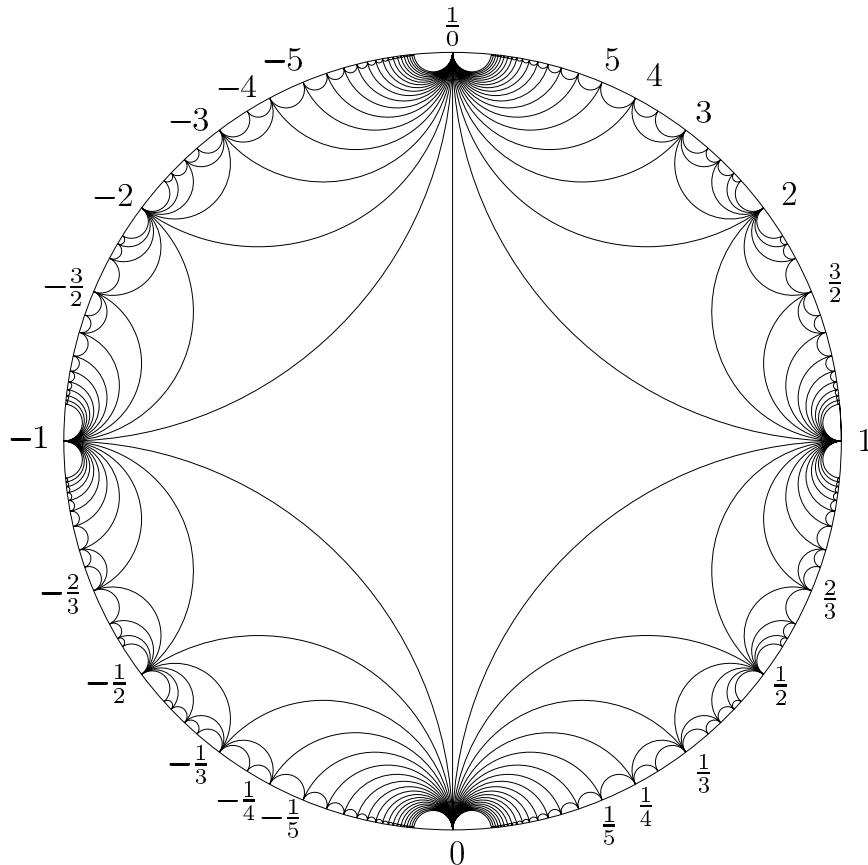


FIGURE 1. The Farey graph in the unit disk, with some of the vertices labelled

traces $x = \text{tr}(\tilde{\rho}(\alpha))$, $y = \text{tr}(\tilde{\rho}(\beta))$, $z = \text{tr}(\tilde{\rho}(\gamma))$, they satisfy the Markoff equation

$$(1) \quad x^2 + y^2 + z^2 = xyz.$$

Moreover, if $\delta \in \mathcal{C}(\Sigma)$ is adjacent to α, β with trace $w = \text{tr}(\tilde{\rho}(\delta))$, then these satisfy the equation

$$(2) \quad w + z = xy.$$

These formulae follow from trace identities in $\text{SL}_2\mathbb{C}$, see [Bow98] for details.

Let $\mathbb{D} = \overline{\mathbb{H}^2}$. Bonahon showed the existence of end invariants $(\nu_-, \nu_+) \subset (\mathbb{D} \times \mathbb{D}) \setminus \Delta$, where Δ is the diagonal of $\partial\mathbb{D} \times \partial\mathbb{D}$ [Bon86]. Teichmüller space for the once punctured torus is identified with $\text{int } \mathbb{D} = \mathbb{H}^2$. Minsky proved that the end invariants of G determine G uniquely up to conjugacy in $\text{PSL}_2\mathbb{C}$ (Theorem A, [M99]). We may thus call ν_+ a *transcendental ending lamination* if the punctured torus group associated to (ν_-, ν_+) is transcendental for all $\nu_- \in \mathbb{D} - \{\nu_+\}$.

Let N be a punctured torus manifold. If one removes the interior of the Margulis tube associated to the parabolic commutator to get \hat{N} , then \hat{N} has two ends, e_{\pm} . The ends are either geometrically finite, or simply degenerate. To each end, associate an

end invariant $\nu_{\pm} \in \mathbb{D}$. If e_s ($s \in \{+, -\}$) is geometrically finite, then $\nu_s \in \text{int } \mathbb{D}$ is the point in Teichmüller space of the conformal structure of the domain of discontinuity corresponding to e_s , or if there is an accidental parabolic, it is the curve $\nu_s \in \hat{\mathbb{Q}} = \mathcal{C}(\Sigma)$ represented by the accidental parabolic in e_s . If e_s is simply degenerate, then there is a unique ending lamination $\nu_s \in \mathbb{R} - \mathbb{Q}$ which is a limit point of simple curves in $\mathcal{C}(\Sigma)$ whose geodesic representatives in N have bounded length and exit the end e_s .

To each end invariant ν_s , we associate an element α_s of $\hat{\mathbb{R}}$, $s = \pm$. If the end e_s is simply degenerate or has an accidental parabolic, then $\alpha_s = \nu_s$. If e_s is geometrically finite, then α_s is a systole in the hyperbolic structure on Σ corresponding to the Teichmüller point ν_s , which is defined up to finite ambiguity. We will assume from now on that e_+ is simply degenerate, so that $\nu_+ \in \mathbb{R} - \mathbb{Q}$. Define $E = E(\alpha_-, \alpha_+)$ to be the set of edges of the Farey graph which separate α_- from α_+ in \mathbb{D} . Let $P_0 \in \hat{\mathbb{Q}}$ be the set of vertices of $\mathcal{C}(\Sigma)$ belonging to at least two edges of E . The edges of E admit a natural order where $e < f$ if e separates the interior of f from α_- , and this induces an order on P_0 (see figure 2). So we can arrange P_0 as a sequence $\{\alpha_n\}_{n=\iota}^{\infty}$, where $\iota = -\infty$ if $\nu_- \in \mathbb{R} \setminus \mathbb{Q}$, and $\iota = 0$ otherwise. We note that the vertices of E which are not elements of P_0 also are ordered, and we will denote these by $\{\beta_i\}_{i=\iota}^{\infty}$, with the convention that β_1 is adjacent to α_0 in $\mathcal{C}(\Sigma)$ and $\alpha_0 < \beta_1$ (meaning that any edge of E incident with α_0 is $<$ the unique edge of E incident with β_1). The *width* of α_i is $1 +$ the number of vertices in $E \setminus P_0$ which are adjacent to α_i in $\mathcal{C}(\Sigma)$ and which are $> \alpha_{i-1}$ and $< \alpha_{i+1}$ (in the example in figure 2, $w(0) = 3, w(1) = 4, w(2) = 2, w(3) = 1, w(4) = 3$).

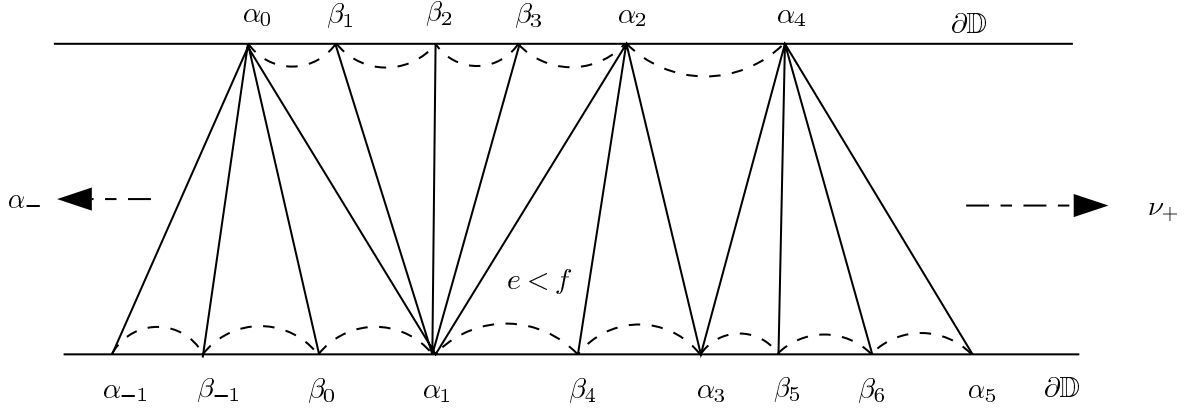


FIGURE 2. The pivot sequence

The pivot sequence $\{\alpha_n\}_{n=0}^{\infty}$ determines ν_+ uniquely. In fact, given adjacent $\alpha_0, \alpha_1 \in \mathcal{C}(\Sigma)$, and the width sequence $\{w(n)\}_{n \in \mathbb{N}}$, the pivot sequence $\{\alpha_n\}$ and ν_+ are uniquely determined. Assuming a fixed lift of a punctured torus group $\tilde{\rho} : \pi_1 \Sigma \rightarrow \text{SL}_2 \mathbb{C}$, let us denote $a_i = \text{tr}(\tilde{\rho}(\alpha_i))$, $b_j = \text{tr}(\tilde{\rho}(\beta_j))$. From the trace relation equation 2, we can compute a_i or b_j recursively as a polynomial function of the traces of $\alpha_0, \alpha_1, \beta_1$. In the example in figure 2, $b_2 = b_1 a_1 - a_0$, $b_3 = b_2 a_1 - b_1 = (b_1 a_1 - a_0) a_1 - b_1$. We let $\lambda(\gamma) = l + i\theta$ be the complex translation length of an element (or conjugacy class)

$\gamma \in \mathrm{PSL}_2\mathbb{C}$, normalized so that $l \geq 0, \theta \in (-\pi, \pi]$. It is determined by the identity $\mathrm{tr}^2\gamma = 4 \cosh^2(\lambda/2)$. Given $\rho : \pi_1(\Sigma) \rightarrow \mathrm{PSL}_2\mathbb{C}$, we get a function on $\mathcal{C}(\Sigma)$ denoted $\lambda(\alpha) \equiv \lambda(\rho(\alpha))$.

3. CONSTRUCTION OF TRANSCENDENTAL ENDING LAMINATIONS

We will describe the construction of width sequences $\{w(n)\}_{n=1}^\infty$ with associated ending lamination ν_+ such that for any choice of ending lamination $\nu_- \in \mathbb{D} - \{\nu_+\}$, the punctured torus group G associated to (ν_-, ν_+) is transcendental. The construction is akin to Cantor diagonalization and to Liouville's construction of transcendental numbers. The non-trivial input which we take is a result of Minsky.

Theorem 3.1. (4.4, 4.5 [M99]) *There exist constants c_2, c_3, c_4 such that*

$$\frac{c_2}{w(n)^2} \leq l(\alpha_n) \leq \frac{c_3}{w(n)^2},$$

$$|w(n) - \frac{2\pi}{\theta(\alpha_n)}| \leq c_4.$$

From this theorem, we clearly have $l(\alpha_n) = O(w(n)^{-2})$, $\theta(\alpha_n) = O(w(n)^{-1})$, so $\lambda(\alpha_n) = O(w(n)^{-1})$. It follows that

$$(3) \quad |a_{n_i}^2 - 4| = |4 \cosh^2(\lambda(\alpha_n)/2) - 4| = O(w(n)^{-2}).$$

Construction: Choose an exhaustion of \mathbb{A} by finite subsets $\mathbb{A} = \bigcup_{i=0}^\infty A_i$, $|A_i| < \infty$, which is possible since \mathbb{A} is countable. Also, choose a sequence $\{n_i\} \subset \mathbb{N}$. Assume by induction that $\{w(1), \dots, w(n_i - 1)\}$ have been defined. We may choose a triple of generators $\alpha_0, \alpha_1, \beta_1$ with traces a_0, a_1, b_1 . Assume that $a_0, a_1, b_1 \subset A_i$. There is a polynomial $P_i(x, y, z) \in \mathbb{Z}[x, y, z]$ such that $a_{n_i} = P_i(a_0, a_1, b_1) \neq \pm 2$ (following from theorem 2.1, $\lambda(\alpha_n) \neq 0$ since any parabolic element is conjugate to either the peripheral curve or to an accidental parabolic corresponding to an end invariant by theorem 4.1 [M99]). Let

$$(4) \quad m_i = \min\{|P_i(x, y, z)^2 - 4| \mid x, y, z \in A_i, P_i(x, y, z) \neq \pm 2\}.$$

Choose $w(n_i)$ large enough that $|a_{n_i}^2 - 4| < m_i$, for any pivot α_{n_i} with width $w(n_i)$, which we may choose by equation 3. We may choose the widths $\{w(n_i+1), \dots, w(n_{i+1}-1)\}$ arbitrarily. This describes the recursive construction of the sequence $\{w(n)\}_{n=1}^\infty$.

Theorem 3.2. *If ν_+ is associated to the width sequence $\{w(1), w(2), \dots\}$ coming from the construction, then for any ending lamination ν_- , the punctured torus group G associated to (ν_-, ν_+) is transcendental.*

Proof. Suppose we had a punctured torus group $G < \mathrm{PSL}_2\mathbb{A}$, with associated ending pair (ν_-, ν_+) , where ν_+ has the width sequence $\{w(n)\}$ given in the construction above. Then $\{a_0, a_1, b_1\} \subset A_i$, for some i . Thus, we have for the pivot α_{n_i} ,

$$0 \neq |a_{n_i}^2 - 2| \geq m_i > |a_{n_i}^2 - 4|,$$

a contradiction. \square

4. CONCLUSION

Most likely, the numbers $\nu_+ \in \mathbb{R} - \mathbb{Q}$ coming from the construction are transcendental, since the continued fraction coefficients of ν_+ are given by $w(n)$ (up to ambiguity of a finite initial sequence), and the numbers $w(n)$ should grow fast enough to satisfy Liouville's criterion for being transcendental. One could get explicit sequences $\{w(n)\}$ which are defined recursively and satisfy the requirements of the construction by using the generalized Liouville theorem, which bounds the length of $P_i(a_0, a_1, b_1)$ in terms of the length of P_i and the lengths of a_0, a_1, b_1 . The length of a polynomial is the sum of the absolute values of its coefficients, and the length of an algebraic number is the length of its minimal polynomial. Then the set $A_i \subset \mathbb{A}$ would be all algebraic numbers with length and degree $\leq i$, which is a finite set, and one could estimate the growth of the bound on $P_i(a_0, a_1, b_1)$ using equation 2 recursively. It might be interesting to compute the asymptotics of $\{w(n)\}$ using this method.

The same technique produces transcendental ending laminations for 4 punctured sphere groups and 2 parabolic generator groups. It would be interesting to try to use the techniques of the general ending lamination theorem to find transcendental ending laminations for arbitrary Kleinian surface groups [M03, BCM].

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